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FINITE ELEMENT APPROXIMATIONS OF AN OPTIMAL CONTROL PROBLEM ASSOCIATED WITH THE SCALAR GINZBURG-LANDAU EQUATION

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Abstract—We consider finite element approximations of an optimal control problem associated with a scalar version of the Ginzburg-Landau equations of superconductivity. The control is the Neumann data on the boundary and the optimization goal is to obtain a best approximation, in the least squares sense, to some desired state. The existence of optimal solutions is proved. The use of Lagrange multipliers is justified and an optimality system of equations is derived. Then, the regularity of solutions of the optimality system is studied, and finally, finite element algorithms are defined and optimal error estimates are obtained.

1. INTRODUCTION

Let Ω be a bounded open domain in $\mathbf{R}^d, d = 2$ or 3 and Γ be the boundary of Ω. We are concerned with the following optimal control problem associated with the scalar Ginzburg-Landau equation: seek a control $g \in L^2(\Gamma)$ and a state $u \in H^1(\Omega)$ such that the functional

$$\mathbf{J}(u,g) = \frac{1}{2}\int_\Omega (u-u_0)^2\,d\Omega + \frac{1}{2}\int_\Gamma g^2\,d\Gamma \tag{1}$$

is minimized subject to the the constraint equations

$$-\Delta u + u^3 - u = 0 \quad \text{in } \Omega \tag{2}$$

and

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma, \tag{3}$$

where u_0 is some prescribed state. We assume that u_0 does not satisfy the constraint equations with $g = 0$; otherwise, $(u_0, 0)$ is a trivial minimizer. The control g is related to an external magnetic field.

The model (2)–(3) is a reduced version of the full Ginzburg-Landau equations valid in the absence of internal magnetic fields; see [8].

In the sequel, we will work with the slightly more general equation

$$-\Delta u + \lambda p u^3 + \lambda\beta(u) = f \quad \text{in } \Omega \tag{4}$$

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$$\frac{\partial u}{\partial n} = g + b \quad \text{on } \Gamma\,, \tag{5}$$

where $p \in C(\bar{\Omega})$ and $\beta \in C^3(\mathbf{R})$. The functions $f \in L^2(\Omega)$ and $b \in L^2(\Gamma)$ are given data. We further assume that there exist positive constants γ_0, $\beta_0 \leq \gamma$, β_1, β_2, β_3 and M such that: $p \geq \gamma > 0$, $|\beta(u)| \leq \beta_0 |u|^3$, $|\beta'(u)| \leq \beta_1 |u|^3$, $|\beta''(u)| \leq \beta_2 |u|^3$ and $|\beta'''(u)| \leq \beta_3 |u|^3$ for $|u| \leq M$. Also, we assume that λ belongs to a compact interval $\Lambda \subset \mathbf{R}_+$ containing 1. Examples for β are $\beta(u) = u$, $\beta(u) = \sin u$, etc.

The weak form of the problem (4)–(5) is given by

$$a(u,v) + \lambda(pu^3, v) + \lambda(\beta(u), v) = (f,v) + (g+b, v)_\Gamma \quad \forall\, v \in H^1(\Omega)\,, \tag{6}$$

where $a(u,v) = \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, d\Omega$ and $(b,v)_\Gamma = \int_\Gamma bv\, d\Gamma$.

The admissible space for the control g is all of $L^2(\Gamma)$. The ***admissibility set*** $\mathbf{U}_{ad}$ is defined by

$$\mathbf{U}_{ad} = \{(u,g) \in H^1(\Omega) \times L^2(\Gamma) : \mathbf{J}(u,g) < \infty, \text{ and (6) is satisfied}\}\,. \tag{7}$$

Then, $(\hat{u}, \hat{g}) \in \mathbf{U}_{ad}$ is called an *optimal solution* if there exists $\epsilon > 0$ such that

$$\mathbf{J}(\hat{u},\hat{g}) \leq \mathbf{J}(u,g) \quad \forall\, (u,g) \in \mathbf{U}_{ad} \text{ satisfying } \quad \|u - \hat{u}\|_1 + \|g - \hat{g}\|_{0,\Gamma} \leq \epsilon\,. \tag{8}$$

2. THE EXISTENCE OF OPTIMAL SOLUTIONS

We first show that an optimal solution exists.

Theorem 2.1 *There exists a* $(\hat{u},\hat{g}) \in \mathbf{U}_{ad}$ *such that* (1) *is minimized.*

Proof: We first claim that $\mathbf{U}_{ad}$, defined by (7), is not empty. From $p \geq \gamma > 0$ and the assumptions on β, we easily see that the term pu^4 dominates the term $u\beta(u)$ for sufficiently large u. Thus

$$\begin{aligned} &\int_\Omega pu^4\, d\Omega + \int_\Omega u\beta(u)\, d\Omega = \int_{|u|>M} (pu^4 + u\beta(u))\, d\Omega + \int_{|u|\leq M} (pu^4 + u\beta(u))\, d\Omega \\ &\geq (\gamma - \beta_0) \int_\Omega u^4\, d\Omega - \int_{|u|\leq M} pu^4\, d\Omega - \int_{|u|\leq M} |u\beta(u)|\, d\Omega \geq (\gamma - \beta_0)\int_\Omega u^4\, d\Omega - L\,, \end{aligned} \tag{9}$$

where $L = |\Omega|\, \|p\|_{L^\infty(\Omega)}\, M^4 - |\Omega|\, M \sup_{|u|\leq M} |\beta(u)|$. Then, we may show, by a standard Galerkin procedure, that there exist a $\tilde{u} \in H^1_0(\Omega)$ such that

$$a(\tilde{u}, v) + \lambda(p\tilde{u}^3, v) + \lambda(\beta(\tilde{u}), v) = (f,v) \quad \forall\, v \in H^1_0(\Omega)\,.$$

Indeed, (9) allows us to derive a uniform bound for the Galerkin sequence; then there is no difficulty to extract subsequences that converge weakly and strongly in appropriate spaces and pass to the limit in the equation. By the regularity results for elliptic equations, we have $\tilde{u} \in H^2(\Omega)$, so that $\partial\tilde{u}/\partial n \in H^{1/2}(\Gamma)$. Now, we set $\tilde{g} = \partial\tilde{u}/\partial n$. Then $(\tilde{u}, \tilde{g})$ satisfy the constraint equations. It is obvious that $\mathbf{J}(\tilde{u},\tilde{g}) < \infty$. Thus, $(\tilde{u},\tilde{g}) \in \mathbf{U}_{ad}$.

Now, let $\{u^{(n)}, g^{(n)}\}$ be a minimizing sequence in $\mathbf{U}_{ad}$ satisfying the equation

$$a(u^{(n)}, v) + \lambda(p(u^{(n)})^3, v) + \lambda(\beta(u^{(n)}), v) = (f,v) + (g^{(n)} + b, v)_\Gamma \quad \forall\, v \in H^1(\Omega)\,. \tag{10}$$

We set $v = u^{(n)}$. Using (9) and the facts that $(f, u^{(n)}) \leq C_\epsilon \|f\|_0^2 + \epsilon \|u^{(n)}\|_1^2$, $(g^{(n)} + b, u^{(n)})_\Gamma \leq C_\epsilon(\|g^{(n)}\|_{0,\Gamma}^2 + \|b\|_{0,\Gamma}^2) + \epsilon\|u^{(n)}\|_{0,\Gamma}^2 \leq C_\epsilon(\|g^{(n)}\|_{0,\Gamma}^2 + \|b\|_{0,\Gamma}^2) + \epsilon C_0 \|u^{(n)}\|_1^2$, and $\int_\Omega (u^{(n)})^2\, d\Omega \leq \int_\Omega ((u^{(n)})^4 + 1)\, d\Omega \leq \int_\Omega (u^{(n)})^4\, d\Omega + |\Omega|$, we obtain:

$$a(u^{(n)}, u^{(n)}) + \lambda(\gamma - \beta_0)\int_\Omega (u^{(n)})^2\, d\Omega - \lambda(\gamma - \beta_0)|\Omega| - L$$

$$\le C_\epsilon \left(\|f\|_0^2 + \|g^{(n)}\|_{0,\Gamma}^2 + \|b\|_{0,\Gamma}^2\right) + \epsilon(C_0+1)\|u^{(n)}\|_1^2$$

so that, by taking $\epsilon = \dfrac{1}{2(C_0+1)}\min\{1, \lambda(\gamma-\beta_0)\}$, we derive:

$$\frac{1}{2}\min\{1, \lambda(\gamma-\beta_0)\}\,\|u^{(n)}\|_1^2 \le C_\epsilon \left(\|f\|_0^2 + \|g^{(n)}\|_{0,\Gamma}^2 + \|b\|_{0,\Gamma}^2\right) + \lambda(\gamma-\beta_0)|\Omega| + L\,.$$

Also, by (1), $\{g^{(n)}\}$ is uniformly bounded in $L^2(\Gamma)$. Thus we have that $\{u^{(n)}\}$ is uniformly bounded in $H^1(\Omega)$. Now we may extract subsequences such that

$$g^{(n)} \rightharpoonup \hat{g} \quad \text{in } L^2(\Gamma)\,,$$

$$u^{(n)} \rightharpoonup \hat{u} \quad \text{in } H^1(\Omega)$$

and

$$u^{(n)} \to \hat{u} \quad \text{in } L^3(\Omega)\,.$$

We may easily pass to the limit in equation (10) to see that $(\hat{u},\hat{g})$ satisfies the constraint equation.

Finally, by the weak lower semicontinuity of $\mathbf{J}(\cdot,\cdot)$, we conclude that $(\hat{u},\hat{g})$ is an optimal solution, i.e.,

$$\mathbf{J}(\hat{u},\hat{g}) = \inf_{(u,g)\in\mathbf{U}_{ad}} \mathbf{J}(u,g)\,. \qquad \blacksquare$$

Remark *Because the optimal control $\hat{g} \in L^2(\Gamma)$, we may deduce, using regularity results for elliptic equations, that $\hat{u} \in H^{3/2}(\Omega)$, provided $b \in L^2(\Gamma)$.*

3. THE EXISTENCE OF LAGRANGE MULTIPLIERS

We wish to use the method of Lagrange multipliers to turn the constrained optimization problem into an unconstrained one. We first show that suitable Lagrange multipliers exist.

Let $B_1 = H^1(\Omega)\times L^2(\Gamma)$ and $B_2 = (H^1(\Omega))^*$, where $(H^1(\Omega))^*$ denotes the dual space of $H^1(\Omega)$. Also, let the nonlinear mapping $M : B_1 \to B_2$ denote the generalized constraint equations, i.e., $M(u,g) = f^*$ for $(u,g)\in B_1$ and $f^*\in B_2$ if and only if

$$a(u,v) + \lambda(pu^3,v) + \lambda(\beta(u),v) - (g,v)_\Gamma = (f^*,v) \quad \forall\, v \in H^1(\Omega)\,.$$

When $f^* \in (H^1(\Omega))^*$ is given by $(f^*,v) = (f,v) + (b,v)_\Gamma$, $\forall\, v \in H^1(\Omega)$, the above reduces to the original constraint equation in the weak form.

In order to justify the use of Lagrange multipliers, we will need the following lemmas, the first of which is due to Georgesco.

Lemma 3.1 *Let $\Omega \subset \mathbf{R}^2$ or $\mathbf{R}^3$ be open and connected. Let $V \in L^q_{\mathrm{loc}}(\Omega)$ for some $q \ge 2$ and $W_1,\ldots,W_n \in L^{2n-1}_{\mathrm{loc}}$. If $\psi \in H^1_{\mathrm{loc}}(\Omega), (-\Delta + \sum_{j=1}^n W_j D_j + V)\psi = 0$ (as a distribution on Ω) and $\psi(x) = 0$ on an open, non-empty subset of Ω, then $\psi = 0$ on Ω.*

Proof: See [6]. $\blacksquare$

Lemma 3.2 *If $w \in H^1(\Omega)$ satisfies*

$$a(w,v) + 3\lambda(p\hat{u}^2 w,v) + \lambda(\beta'(\hat{u})w,v) = 0 \quad \forall\, v \in H^1(\Omega) \tag{11}$$

and $w = 0$ on Γ, then $w = 0$ in Ω.

Proof: Let Ω_e be a smooth extension of the domain Ω through a connected piece of Γ as shown in the figure, i.e., $\Omega_e = \Omega \cup \omega \cup (\partial\Omega \cap \partial\omega)$.

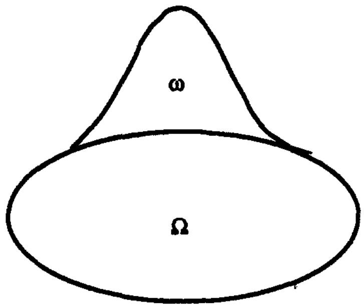

Figure 3.1 - The domains Ω and ω.

We define

$$w_e = \begin{cases} w & \text{in } \Omega \\ 0 & \text{in } \Omega_e \backslash \Omega. \end{cases}$$

Because $w = 0$ on Γ, we have that $w_e \in H^1(\Omega_e)$. We extend $\hat{u}$ by zero outside Ω so that $\hat{u}$ belongs to $L^6(\Omega_e)$. By splitting the volume integrals over Ω_e into that over Ω and ω and using (11), we obtain:

$$\int_{\Omega_e} \mathbf{grad}\, v \cdot \mathbf{grad}\, w_e \, d\Omega + 3\lambda \int_{\Omega_e} p\hat{u}^2 v w_e \, d\Omega + \lambda \int_{\Omega_e} \beta'(\hat{u}) v w_e \, d\Omega$$

$$= 0 \quad \forall\, v \in H_0^1(\Omega_e).$$

This shows that w_e is a solution (in the sense of distributions) of

$$-\Delta w_e + \lambda(3p\hat{u}^2 + \beta'(\hat{u}))w_e = 0 \quad \text{in } \Omega_e .$$

Note that $3p\hat{u}^2 + \beta\hat{u}^{\beta-1} \in L^2(\Omega_e)$ so that, by Georgesco's Lemma, we conclude that $w_e = 0$ in Ω_e, or, $w = 0$ in Ω. ∎

Theorem 3.3 *Let $(\hat{u}, \hat{g}) \in H^1(\Omega) \times L^2(\Gamma)$ denote an optimal solution in the sense of* (8). *Then there exists a nonzero Lagrange multiplier $\xi \in H^1(\Omega)$ satisfying the Euler equation*

$$\mathbf{J}'(\hat{u}, \hat{g}) \cdot (w, z) + < M'(\hat{u}, \hat{g}) \cdot (w, z), \xi > = 0 \quad \forall\, (w, z) \in H^1(\Omega) \times L^2(\Gamma), \tag{12}$$

where $< \cdot, \cdot >$ denotes the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))^*$.

Proof: The operator $M'(\hat{u}, \hat{g}) \in \mathbf{L}(B_1; B_2)$ may be defined as follows: $M'(\hat{u}, \hat{g}) \cdot (w, z) = \bar{f}$ for $(w, z) \in B_1$ and $\bar{f} \in B_2$ if and only if

$$a(w, v) + 3\lambda(p\hat{u}^2 w, v) + \lambda(\beta'(\hat{u})w, v) - (z, v)_\Gamma = (\bar{f}, v) \quad \forall\, v \in H^1(\Omega). \tag{13}$$

The operator $M'(\hat{u}, \hat{g})$ from B_1 into B_2 is onto. To see this, first note that the mapping C from $H^1(\Omega)$ into $(H^1(\Omega))^*$, defined by $Cw = 3\lambda p\hat{u}^2 w + \lambda\beta'(\hat{u})w - w$ for all $w \in H^1(\Omega)$, is compact, since $3\lambda p\hat{u}^2 w + \lambda\beta'(\hat{u})w - w \in L^{3/2}(\Omega)$ and $L^{3/2}(\Omega)$ is compactly imbedded into $(H^1(\Omega))^*$. Consider the operator D from $H^1(\Omega)$ into $(H^1(\Omega))^*$ defined as follows: $D(w) = \bar{f}$ for $w \in H^1(\Omega)$ and $\bar{f} \in (H^1(\Omega))^*$ if and only if

$$a(w, v) + 3\lambda(p\hat{u}^2 w, v) + \lambda(\beta'(\hat{u})w, v) = (\bar{f}, v) \quad \forall\, v \in H^1(\Omega).$$

It follows from the compactness of the operator C that D is a compact perturbation of the operator $-\Delta + I$. Thus by Fredholm alternatives, there are only two possibilities: either D is onto; or, there are a finite number of solutions $\{w_i, i = 1, \ldots, N\}$ to the homogeneous equation

$$a(w, v) + 3\lambda(p\hat{u}^2 w, v) + \lambda(\beta'(\hat{u})w, v) = 0 \quad \forall\, v \in H^1(\Omega),$$

and the equation $D(w) = f^*$ is solvable for a given $f^* \in (H^1(\Omega))^*$ if and only if $< f^*, w_i > = 0, i = 1, \ldots, N$. In the former case, it is obvious that the operator $M'(\hat{u}, \hat{g})$ is onto by setting $z = 0$ in equation (13). In the latter case, since $\{w_i\}$ are eigenfunctions and thus nontrivial in Ω, it follows from Lemma 2.2 that $\|w_i\|_{0,\Gamma} \neq 0$, $i = 1, \ldots, N$. Without loss of generality, we may assume that $\{w_i, i = 1, \ldots, N\}$ are orthonormalized in $L^2(\Gamma)$, i.e., $(w_i, w_j)_\Gamma = \delta_{i,j}$. For any given $\bar{f} \in H^1(\Omega)^*$, we define $f^* \in H^1(\Omega)^*$ by $< f^*, v > = (\bar{f}, v) + (z, v)_\Gamma$ for $\forall\, v \in H^1(\Omega)$ where $z = \sum_{i=1}^N (\bar{f}, w_i) w_i$ so that $< f^*, w_i > = 0, i = 1, \ldots, N$. Then there exists a $w \in H^1(\Omega)$ such that $D(w) = f^*$, i.e.,

$$a(w, v) + 3\lambda(p\hat{u}^2 w, v) + \lambda(\beta'(\hat{u}) w, v) - (z, v)_\Gamma = (\bar{f}, v) \quad \forall\, v \in H^1(\Omega).$$

Thus we have shown in both cases that the operator $M'(\hat{u}, \hat{g})$ is onto.

Now consider the nonlinear operator $N : B_1 \to \mathbb{R} \times B_2$ defined by

$$N(u, g) = \begin{pmatrix} \mathbf{J}(u, g) - \mathbf{J}(\hat{u}, \hat{g}) \\ M(u, g) \end{pmatrix}.$$

The operator $N'(\hat{u}, \hat{g})$ from B_1 into $\mathbb{R} \times B_2$ may be defined as follows: $N'(\hat{u}, \hat{g}) \cdot (w, z) = (\tilde{r}, \tilde{f})$ for $(w, z) \in B_1$ and $(\tilde{r}, \tilde{f}) \in \mathbb{R} \times B_2$ if and only if

$$(\hat{u} - u_0, w) + (\hat{g}, z) = \tilde{r}$$

and

$$a(w, v) + 3\lambda(p\hat{u}^2 w, v) + \lambda(\beta'(\hat{u}) w, v) - (z, v)_\Gamma = (\tilde{f}, v) \quad \forall\, v \in H^1(\Omega).$$

This operator has a closed range but is not onto. The fact that it has a closed range follows easily from the fact that the operator $M'(\hat{u}, \hat{g})$ is onto and the following well-known result. Let X, Y, Z be Banach spaces and $A : X \to Y$ and $B : X \to Z$ be linear continuous operators. Then, if the range of A is closed in Y and the subspace $B \ker(A)$ is closed in Z and if, further, $Cx = (Ax, Bx), C : X \to Y \times Z$, then the range of C is closed in $Y \times Z$. Thus, in our context, the operator $N'(\hat{u}, \hat{g})$ has a closed range in B_2.

The operator $N'(\hat{u}, \hat{g})$ is not onto because if it were, by the Implicit Function Theorem, we would have $(\tilde{u}, \tilde{g}) \in \mathbf{U}_{ad}$ such that $\|\hat{u} - \tilde{u}\|_1 + \|\hat{g} - \tilde{g}\|_{0,\Gamma} \leq \epsilon$, $\mathbf{J}(\tilde{u}, \tilde{g}) < \mathbf{J}(\hat{u}, \hat{g})$, and the constraint equations are satisfied, contradicting the hypothesis that $(\hat{u}, \hat{g})$ is an optimal solution. Then, the Hahn-Banach theorem implies that there exists a nonzero element of $(\mathbb{R} \times B_2)^* = \mathbb{R} \times H^1(\Omega)$ that annihilates the range of $N'(\hat{u}, \hat{g})$, *i.e.*, there exists $(r, \xi) \in \mathbb{R} \times H^1(\Omega)$ such that

$$< (\tilde{r}, \tilde{f}), (r, \xi) > = 0 \quad \forall\, (\tilde{r}, \tilde{f}) \text{ in the range of } N'(\hat{u}, \hat{g}), \tag{14}$$

where $< \cdot, \cdot >$ denotes the duality pairing between $\mathbb{R} \times B_2$ and its dual $(\mathbb{R} \times B_2)^*$. Note that $r \neq 0$ since otherwise we would have that $< \tilde{f}, \xi > = 0$ for all $\tilde{f} \in B_2$ so that $\xi \equiv 0$, contradicting the fact that $(r, \xi) \neq 0$. We may, without any loss of generality, set $r = -1$. Clearly, using the definition of the operator $N'(\hat{u}, \hat{g})$, (12) and (14) are equivalent. ∎

Equation (12) may be rewritten in the form

$$-(\hat{u} - u_0, w) - (\hat{g}, z)_\Gamma + a(w, \xi) + 3\lambda(p\hat{u}^2 w, \xi) + \lambda(\beta'(\hat{u}) w, \xi) - (z, \xi)_\Gamma$$

$$= 0 \quad \forall\, (w, z) \in H^1(\Omega) \times L^2(\Gamma).$$

Upon separating the above equations, we have that

$$a(w, \xi) + 3\lambda(p\hat{u}^2 w, \xi) + \lambda(q\beta'(\hat{u}) w, \xi) = (\hat{u} - u_0, w) \quad \forall\, w \in H^1(\Omega) \tag{15}$$

and

$$(\hat{g}, z) = -(z, \xi) \quad \forall\, z \in L^2(\Gamma). \tag{16}$$

4. AN OPTIMALITY SYSTEM AND THE REGULARITY OF ITS SOLUTIONS

From (15)–(16) and the original constraint equation (6), we form the following system of equations (dispensing with the hat notations to denote optimal solutions):

$$a(u,v)+\lambda(pu^3,v)+\lambda(\beta(u),v)=(f,v)-(\xi,v)_\Gamma+(b,v)_\Gamma \quad \forall\, v\in H^1(\Omega) \tag{17}$$

and

$$a(w,\xi)+3\lambda(pu^2w,\xi)+\lambda(\beta'(u)w,\xi)=(u-u_0,w) \quad \forall\, w\in H^1(\Omega)\,. \tag{18}$$

This system of equations will be called the *optimality system.*

Integrations by parts may be used to show that the system (17)–(18) constitutes a weak formulation of the problem

$$-\Delta u+\lambda pu^3+\lambda\beta(u)=f \quad \text{in } \Omega\,, \tag{19}$$

$$\frac{\partial u}{\partial n}=b-\xi \quad \text{on } \Gamma\,, \tag{20}$$

$$-\Delta\xi+\lambda(3pu^2+\beta'(u))\xi=u-u_0 \quad \text{in } \Omega \tag{21}$$

and

$$\frac{\partial\xi}{\partial n}=0 \quad \text{on } \Gamma\,. \tag{22}$$

Now we examine the regularity of solutions of the optimality system (17)–(18), or equivalently, (19)–(22).

Theorem 4.1 *Suppose the given data satisfies $b\in H^{1/2}(\Gamma)$, $f\in L^2(\Omega)$ and $u_0\in L^2(\Omega)$. Suppose that Ω is of class $C^{1,1}$. Then, if $(u,\xi)\in H^1(\Omega)\times H^1(\Omega)$ denotes a solution of the optimality system* (17)–(18), *or equivalently,* (19)–(22), *then we have that $(u,\xi)\in H^2(\Omega)\times H^2(\Omega)$.*

Proof: Since $u,\xi\in H^1(\Omega)$, using Sobolev imbedding results and growth conditions for β and β', we deduce that $\lambda pu^3+\lambda\beta(u)\in L^2(\Omega)$ and $\lambda(3pu^2\xi+\beta\beta'(u))\xi\in L^2(\Omega)$. Also, from the fact that $\xi\in H^{1/2}(\Gamma)$, we see that $\partial u/\partial n\in H^{1/2}(\Gamma)$. Thus by applying elliptic regularity results to equations (19)–(20) and (21)–(22), respectively, we obtain $u\in H^2(\Omega)$ and $\xi\in H^2(\Omega)$. ∎

Remark *The above result also holds for convex regions of $\mathbf{R}^2$. In general, we may show by bootstrap techniques, that if $f\in H^m(\Omega)$, $u_0\in H^m(\Omega)$, $b\in H^{m+1/2}(\Gamma)$ and p,q and Ω are sufficiently smooth, then $(u,\xi)\in H^{m+2}(\Omega)\times H^{m+2}(\Omega)$. In particular, if f,u_0,p,q and b are all of class $C^\infty(\overline{\Omega})$ and Ω is of class C^∞, then u and ξ are $C^\infty(\overline{\Omega})$ functions as well.*

5. FINITE ELEMENT APPROXIMATIONS

A finite element discretization of the optimality system (17)–(18) is defined in the usual manner. We first choose families of finite dimensional subspaces $V^h\subset H^1(\Omega)$ satisfying the approximation property: there exists a constant C and an integer k such that

$$\|v-v^h\|_1\le Ch^m\|v\|_{m+1}\,, \quad \forall\, v\in H^{m+1}(\Omega)\,,\ 1\le m\le k\,. \tag{23}$$

One may consult, e.g., [2] or [4] for a catalogue of finite element spaces satisfying (23). Then, we may formulate the approximate problem for the optimality system (17)–(18): seek $u^h\in V^h$ and $\xi^h\in V^h$ such that

$$a(u^h,v^h)+\lambda(p(u^h)^3,v^h)+\lambda(\beta(u^h),v^h)=(f,v^h)-(\xi^h,v^h)_\Gamma+(b,v^h)_\Gamma \quad \forall\, v^h\in V^h \tag{24}$$

and

$$a(\xi^h,w^h)+3\lambda(p(u^h)^2\xi^h,w^h)+\beta\lambda(\beta'(u^h)\xi^h,w^h)=(u^h-u_0,w^h) \quad \forall\, w^h\in V^h\,. \tag{25}$$

In order to derive error estimates, we begin by recasting the optimality system (17)–(18) and its discretization (24)–(25) into a form that fits into the framework of *Brezzi-Rappaz-Raviart* theory concerning the approximation of a class of nonlinear problems; see [3], [5] and [7].

We define

$$X = H^1(\Omega) \times H^1(\Omega),$$

$$Y = H^1(\Omega)^* \times H^{-1/2}(\Gamma) \times H^1(\Omega)^*,$$

$$Z = L^{3/2}(\Omega) \times L^2(\Gamma) \times L^{3/2}(\Omega)$$

and

$$X^h = V^h \times V^h$$

where $H^1(\Omega)^*$ denotes the dual space of $H^1(\Omega)$. Note that $Z \subset Y$ with a compact imbedding.

Let the operator $T \in \mathbf{L}(Y;X)$ be defined in the following manner: $T(\zeta,\theta,\eta) = (u,\xi)$ for $(\zeta,\theta,\eta) \in Y$ and $(u,\xi) \in X$ if and only if

$$a(u,v) + (u,v) = (\zeta,v) + (\theta,v)_\Gamma \quad \forall\, v \in H^1(\Omega) \tag{26}$$

and

$$a(\xi,\omega) + (\xi,\omega) = (\eta,\omega) \quad \forall\, \omega \in H^1(\Omega). \tag{27}$$

Clearly, (26)–(27) consists of two *uncoupled* Poisson-type equations and T is its solution operator.

Analogously, the operator $T^h \in \mathbf{L}(Y;X)$ is defined as follows: $T^h(\zeta,\theta,\eta) = (u^h,\xi^h)$ for $(\zeta,\theta,\eta) \in Y$ and $(u^h,\xi^h) \in X^h$ if and only if

$$a(u^h,v^h) + (u^h,v^h) = (\zeta,v^h) + (\theta,v^h)_\Gamma \quad \forall\, v^h \in V^h \tag{28}$$

and

$$a(\xi^h,\omega^h) + (\xi^h,\omega^h) = (\eta,\omega^h) \quad \forall\, \omega^h \in V^h. \tag{29}$$

Clearly, (28)–(29) consists of two discrete Poisson-type equations that are discretizations of the equations (26)–(27); also, T^h is the solution operator for these two discrete equations.

By the well-known results concerning the approximation of elliptic equations (see, e.g., [2] or [4]), we obtain:

$$\|(T - T^h)(\zeta,\theta,\eta)\|_X \to 0$$

as $h \to 0$, for all $(\zeta,\theta,\eta) \in Y$. Also, because $Z \subset Y$ with a compact imbedding, we have that

$$\|(T - T^h)\|_{\mathbf{L}(Z;X)} \to 0$$

as $h \to 0$.

Let Λ denote a compact subset of $\mathbf{R}_+$ containing 1. Next, we define the *nonlinear* mapping $G : \Lambda \times X \to Y$ as follows: $G(\lambda,(u,\xi)) = (\zeta,\eta)$ for $\lambda \in \Lambda$, $(u,\xi) \in X$ and $(\zeta,\theta,\eta) \in Y$ if and only if

$$(\zeta,v) = \lambda(pu^3 + \beta(u) - u, v) - (f,v) \quad \forall\, v \in H^1(\Omega), \tag{30}$$

$$(\theta,w)_\Gamma = -(b - \xi, w)_\Gamma \quad \forall\, w \in H^1(\Omega) \tag{31}$$

and

$$(\eta,\omega) = \lambda(3pu^2\xi + \beta'(u)\xi - \xi, \omega) - (u - u_0, \omega) \quad \forall\, \omega \in H^1(\Omega). \tag{32}$$

It is easily seen that the optimality system (17)–(18) is equivalent to

$$(u,\xi) + TG(\lambda,(u,\xi)) = 0 \tag{33}$$

and that the discrete optimality system (24)–(25) is equivalent to

$$(u^h,\xi^h) + T^h G(\lambda,(u^h,\xi^h)) = 0. \tag{34}$$

We have thus recast our continuous and discrete optimality problems into a form that enables us to apply the theories of [3], [5] and [7].

By differentiating (30)–(32), the operator $D_\phi G$, the derivative of G with respect to (u,ξ), may be defined as follows. For given $\lambda \in \Lambda$ and $(u,\xi) \in X$, $D_\phi G(\lambda,(u,\xi))(v,\omega) = (\tilde{\zeta},\tilde{\theta},\tilde{\eta})$ for $(v,\omega)\in X$ if and only if

$$(\tilde{\zeta},\bar{v}) = \lambda(3pu^2v + \beta'(u)v - v,\bar{v}) \quad \forall\, \bar{v} \in H^1(\Omega)\,,$$

$$(\tilde{\theta},\bar{z})_\Gamma = (\omega,\bar{z})_\Gamma \quad \forall\, \bar{z} \in H^1(\Omega)$$

and

$$(\tilde{\eta},\bar{\omega}) = \lambda(3pu^2\omega + 6puv\xi + \beta''(u)v\xi + \beta(u)\omega - \omega,\bar{\omega}) - (v,\bar{\omega}) \quad \forall\, \omega \in H^1(\Omega)\,.$$

A solution $(u(\lambda),\xi(\lambda))$ of the problem (17)–(18), or equivalently, of (33), is nonsingular if the linear system

$$a(\tilde{u},v) + \lambda(3pu^2\tilde{u} + \beta'(u)\tilde{u},v) + (\tilde{\xi},v)_\Gamma = (\hat{f},v) \quad \forall\, v \in H^1(\Omega)$$

and

$$a(\tilde{\xi},\omega) - \lambda(6pu\xi\tilde{u} + 3pu^2\tilde{\xi} + \beta''(u)\xi\tilde{u} + \beta'(u)\tilde{\xi},\omega) = (\hat{\eta},\omega) \quad \forall\, \omega \in H^1(\Omega)$$

has a unique solution $(\tilde{u},\tilde{\xi}) \in X$ for every $\hat{f},\hat{\eta} \in H^{-1}(\Omega)$.

An analogous definition holds for nonsingular solutions of the discrete optimality system (24)–(25), or equivalently, (34).

Using the *Brezzi-Rappaz-Raviart* theory, we are led to the following result.

Theorem 5.1 *Assume that Λ is a compact interval of $\mathbb{R}_+$ containing* 1 *and that there exists a branch $\{(\lambda,\phi(\lambda) = (u,\xi)) : \lambda \in \Lambda\}$ of nonsingular solutions of the optimality system* (17) *and* (18). *Assume that the finite element spaces V^h satisfy the conditions* (23). *Then, there exists a neighborhood* O *of the origin in $X = H^1(\Omega) \times H^1(\Omega)$ and, for $h \le h_0$ small enough, a unique branch $\{(\lambda,\phi^h(\lambda) = (u^h,\xi^h)) : \lambda \in \Lambda\}$ of solutions of the discrete optimality system* (24)–(25) *such that $\phi^h(\lambda) - \phi(\lambda) \in \mathrm{O}$ for all $\lambda \in \Lambda$. Moreover,*

$$\|\phi^h(\lambda) - \phi(\lambda)\|_X = \|u(\lambda) - u^h(\lambda)\|_1 + \|\xi(\lambda) - \xi^h(\lambda)\|_1 \to 0 \tag{35}$$

as $h \to 0$, uniformly in $\lambda \in \Lambda$.

If, in addition, the solution of the optimality system satisfies $(u(\lambda),\xi(\lambda)) \in H^{m+1}(\Omega) \times H^{m+1}(\Omega)$ for $\lambda \in \Lambda$, then there exists a constant C, independent of h, such that

$$\|u(\lambda) - u^h(\lambda)\|_1 + \|\xi(\lambda) - \xi^h(\lambda)\|_1 \le Ch^m\big(\|u(\lambda)\|_{m+1} + \|\xi(\lambda)\|_{m+1}\big)\,, \tag{36}$$

uniformly in $\lambda \in \Lambda$. ∎

Again, using the *Brezzi-Rappaz-Raviart* theory, we can derive an estimate for the error of u^h and ξ^h in the $L^2(\Omega)$-norm. We introduce the spaces $H = L^2(\Omega) \times L^2(\Omega)$ and $W = H^2(\Omega) \times H^2(\Omega)$.

Theorem 5.2 *Assume the hypotheses of Theorems* 4.1 *and* 5.1. *Then, for $h \le h_1$ sufficiently small, there exists a constant C, independent of h such that*

$$\|u(\lambda) - u^h(\lambda)\|_0 + \|\xi(\lambda) - \xi^h(\lambda)\|_0 \le Ch^{m+1}\big(\|u(\lambda)\|_{m+1} + \|\xi(\lambda)\|_{m+1}\big)\,. \tag{40}$$ ∎

The proofs of Theorems 5.1 and 5.2 rest in verifying that the requirements in *Brezzi-Rappaz-Raviart* theory hold in our setting. This task is somewhat standard, thus the details are omitted here. The assumptions on the function β are needed in order to fulfil the requirements. We point out that in showing Theorem 5.2, we have used the the L^2-error estimate for second order elliptic equations and the following inequalities:

$$\int_\Omega u^2 vw\, d\Omega \le \|u\|_{L^4(\Omega)} \|v\|_{L^2(\Omega)} \|w\|_{L^4(\Omega)} \le C\|u\|_1 \|v\|_{L^2(\Omega)} \|w\|_1$$

and

$$\begin{aligned}\int_\Omega |\beta(u)vw|\, d\Omega &\le \sup_{|x|\le \|u\|_{L^\infty(\Omega)}} |\beta(x)|\, \|v\|_{L^2(\Omega)} \|w\|_1 \\ &\le \sup_{|x|\le C\|u\|_2} |\beta(x)|\, \|v\|_{L^2(\Omega)} \|w\|_1\end{aligned}$$

for all $u \in H^2(\Omega)$, $v \in L^2(\Omega)$ and $w \in H^1(\Omega)$.

A consequence of Theorems 5.1 and 5.2 is the following corollary that gives estimates for the error in the approximation of the controls.

Corollary 5.3 *Define the approximate control by*

$$g^h = -\xi^h \quad \text{on } \Gamma$$

and assume the hypotheses of Theorem 5.1. *Then, there exists a constant C, independent of h such that for $h \le h_0$ sufficiently small,*

$$\|g^h - g\|_{1/2,\Gamma} \le Ch^m \left(\|u(\lambda)\|_{m+1} + \|\xi(\lambda)\|_{m+1}\right). \tag{37}$$

If in addition the hypotheses of Theorem 5.2 *hold, then, for $h \le h_1$ sufficiently small, there exists a constant C, independent of h such that*

$$\|g^h - g\|_{0,\Gamma} \le Ch^{m+1/2} \left(\|u(\lambda)\|_{m+1} + \|\xi(\lambda)\|_{m+1}\right). \tag{38}$$

Proof: Recall that $g = -\xi$ on Γ; see (16). Then (37) easily follows from (35) and the trace theorem (see [1]), *i.e.*, $\|g - g^h\|_{1/2,\Gamma} \le C\|\xi - \xi^h\|_1$. The estimate (38) follows from (35), (36) and the inequalities (see [1] or [2]) $\|\xi - \xi^h\|_{1/2} \le C\|\xi - \xi^h\|_1 \|\xi - \xi^h\|_0$ and $\|g - g^h\|_{0,\Gamma} \le C\|\xi - \xi^h\|_{1/2}$.
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